

Development and Application of a New Generalization of Interval Numbers in Multi-Criteria Decision Making

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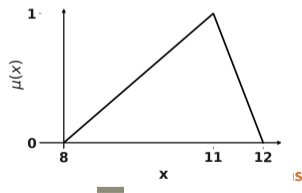
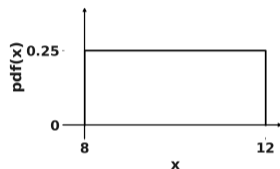
Agenda

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- 3 Asymmetric Interval Number (AIN)
 - Definitions
 - Arithmetic Operations
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Motivation

Suppose that we consider the issue of estimating the time required to complete a project.

- Expert: "It typically takes 8 to 12 days, and on average, it takes about 11."
- Classical Interval: $[8, 12]$
 - only gives lower and upper bounds,
 - implicitly assumes equal probability for early and late outcomes.
- Triangular Fuzzy Number: $(8, 11, 12)$
 - uses a membership function $\mu(x) \in [0, 1]$,
 - describes **possibility**, not probability,
 - identifies 11 as the "most representative" value,
 - but does not quantify how likely early vs. late completion is.



Motivation – Multiplication of Uncertain Quantities

Let us consider a simple symmetric case:

$$X = [0.5, 1.5] \quad \text{or} \quad X = (0.5, 1.0, 1.5)$$

Classical Interval Numbers:

$$X \cdot X = [0.25, 2.25]$$

- Computed only from the extreme points.
- Midpoint 1.25 is often (wrongly) taken as “typical”.
- In reality, most values of X^2 are concentrated near 1.
- True expected value: $E[X^2] = 1.083... \neq 1.25$.

Why is the expected value $E[X^2] = 1.083\dots$?

Let $X \sim U(0.5, 1.5)$.

- PDF of X :

$$f_X(x) = \frac{1}{1.5 - 0.5} = 1, \quad x \in [0.5, 1.5].$$

- Expected value of $Y = X^2$:

$$E[X^2] = \int_{0.5}^{1.5} x^2 f_X(x) dx = \int_{0.5}^{1.5} x^2 dx.$$

- Compute:

$$\int x^2 dx = \frac{x^3}{3} \Rightarrow E[X^2] = \frac{1.5^3 - 0.5^3}{3}.$$

- Numerically:

$$E[X^2] = \frac{3.375 - 0.125}{3} = \frac{3.25}{3} \approx 1.083\dots$$



Motivation – Multiplication of Uncertain Quantities

Triangular Fuzzy Number:

$$X \cdot X = (0.25, 1.0, 2.25)$$

- Identifies 1.0 as “most representative”.
- But describes **possibility**, not probability.
- No information on how likely 0.25 vs. 2.25 is.

Observation

- Classical Interval Numbers (CIN): produce a range $[0.25, 2.25]$ and suggest 1.25 as a “typical” value, which is incorrect since the true expectation is 1.083... (closer to 1).
- Triangular Fuzzy Numbers (TFN): emphasize 1.0 as the central value, but this is not the probabilistic expected value, only a *possibility-based* indication without information on how likely 0.25 or 2.25 are.

AIN definition

Definition

The *AIN* is defined as a closed interval number $[a, b]$ with a given characteristic value c . The formal notation of *AIN* takes the following Eq. (1):

$$X = [a, b]_c = \{x \in R \mid a \leq x \leq b \wedge c = E(X)\} \quad (1)$$

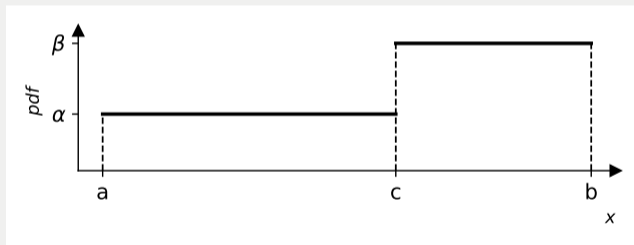
where a , b and c are the real numbers; $E(X)$ is expected value.

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \quad (2)$$

$$E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)dx$$

The approximated probability density function for $AIN [a, b]_c$

$$pdf(x) = \begin{cases} \alpha, & a \leq x < c \\ \beta, & c \leq x \leq b \\ 0, & \text{others} \end{cases} \quad (4)$$



$$\alpha = \frac{b - c}{(b - a)(c - a)}; \quad \beta = \frac{c - a}{(b - a)(b - c)} \quad (5)$$

Cumulative Distribution Function

$$cdf(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx \quad (6)$$

$$cdf(x) = \begin{cases} 0, & x \leq a \\ \alpha(x - a), & a \leq x \leq c \\ \alpha(c - a) + \beta(x - c), & c \leq x \leq b \\ 1, & b \leq x \end{cases} \quad (7)$$

Inverse Cumulative Distribution Function

$$Q(p) = \begin{cases} a + \frac{p}{\alpha}, & 0 \leq p \leq \alpha \cdot (c - a) \\ c + \frac{p - \alpha \cdot (c - a)}{\beta}, & \alpha \cdot (c - a) \leq p \leq 1 \end{cases} \quad (8)$$

Additive and Multiplicative Inverses of AIN

Additive Inverse

For $X = [a, b]_c$:

$$-X = [-b, -a]_{-c}$$

Multiplicative Inverse

We use the **Law of the Unconscious Statistician (LOTUS)**: $E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x) dx$
If $0 \notin [a, b]$:

$$\frac{1}{X} = \left[\frac{1}{b}, \frac{1}{a} \right]_{c_1}$$
$$c_1 = \begin{cases} \alpha \ln\left(\frac{c}{a}\right) + \beta \ln\left(\frac{b}{c}\right), & a \neq b, \\ \frac{1}{a}, & a = b \end{cases}$$

Basic Arithmetic Operations on AIN

Addition / Subtraction

$$X + Y = [a_1 + a_2, b_1 + b_2]_{c_1+c_2}$$

$$X - Y = [a_1 - b_2, b_1 - a_2]_{c_1-c_2}$$

Multiplication

$$X \cdot Y = \left[\min\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}, \max\{a_1a_2, a_1b_2, b_1a_2, b_1b_2\} \right]_{c_1c_2}$$

Division of AIN

Division (if $0 \notin [a_2, b_2]$)

$$X \div Y = \left[\min \left\{ \frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2} \right\}, \max \left\{ \frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2} \right\} \right]_{c_3}$$

$$c_3 = \begin{cases} c_1 \cdot \left(\alpha \ln \left(\frac{c_2}{a_2} \right) + \beta \ln \left(\frac{b_2}{c_2} \right) \right), & a_2 \neq b_2, \\ c_1 \div c_2, & a_2 = b_2 \end{cases}$$

Square Root (for $a \geq 0$)

$$\sqrt{[a, b]_c} = [\sqrt{a}, \sqrt{b}]_{c_2}$$

$$c_2 = \begin{cases} \frac{2}{3}(\alpha(c^{1.5} - a^{1.5}) + \beta(b^{1.5} - c^{1.5})), & a \neq b, \\ \sqrt{c}, & a = b \end{cases}$$

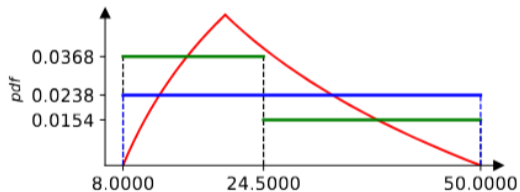
Exponentiation ($n \neq -1$)

$$([a, b]_c)^n = [a^n, b^n]_{c_2}$$

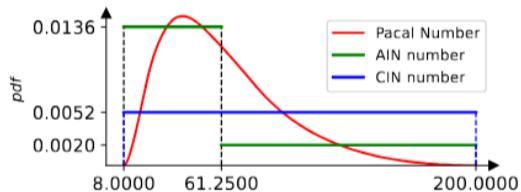
$$c_2 = \begin{cases} \frac{1}{n+1}(\alpha(c^{n+1} - a^{n+1}) + \beta(b^{n+1} - c^{n+1})), & a \neq b, \\ c^n, & a = b \end{cases}$$

Operations that break symmetry?

(a) Result of $[4, 10] \cdot [2, 5]$ where $A = 0.2143$



(b) Result of $([4, 10] \cdot [2, 5]) \div [1, 4]$ where $A = 0.4453$

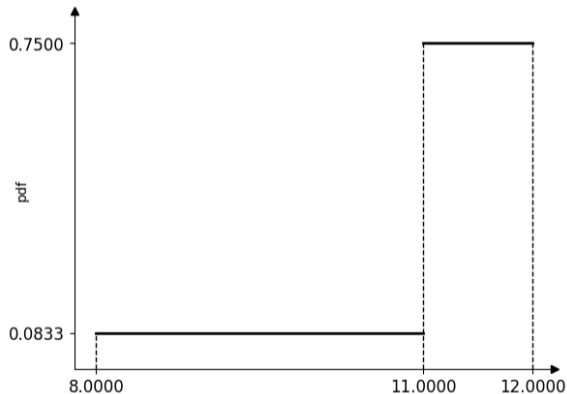


"It typically takes 8 to 12 days, and on average, it takes about 11."

```

=== AIN =====
[8.0000, 12.0000]_{11.0000}
=== Summary =====
Alpha      =      0.083333
Beta       =      0.750000
Assymetry  =     -0.500000
Exp. val.  =     11.000000
Variance   =      1.000000
Std. dev.  =      1.000000
Midpoint   =     10.000000
=====

```



$$[0.5, 1.5]_{1.0} \cdot [0.5, 1.5]_{1.0} = \dots$$

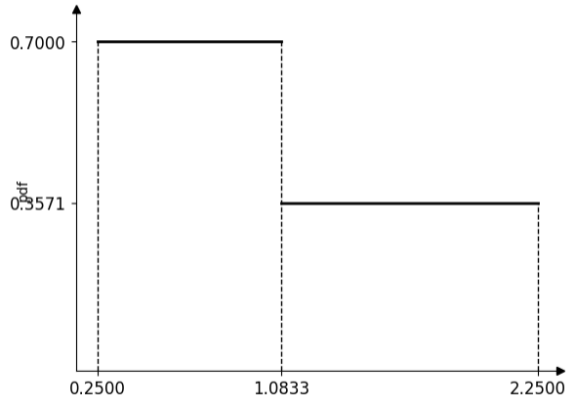
=== AIN =====

$[0.2500, 2.2500]_{\{1.0833\}}$

=== Summary =====

Alpha	=	0.700000
Beta	=	0.357143
Assymetry	=	0.166667
Exp. val.	=	1.083333
Variance	=	0.324074
Std. dev.	=	0.569275
Midpoint	=	1.250000

=====



 or telecommunications

Fuel Consumption on a Small Data Set

- Total fuel consumption over 5 route segments.
- Data: distances (D) and fuel consumption (FC) for each segment (see table).
- Goal: compute total average consumption (TAC) in liters/100 km.
- True reference value: $TAC = 5.9074$.

No.	Distance (D)	Fuel Consumed (FC)	Average (AC)
1	693.0	43	6.2
2	735.8	39	5.3
3	553.6	31	5.6
4	492.5	34	6.9
5	741.4	43	5.8
Σ	3216.3	190	-

Comparison of CIN and AIN Approaches

- Using CIN:

$$D = [492.50, 741.40], \quad FC = [31.00, 43.00]$$

$$TAC = [4.1813, 8.7310], \quad \text{midpoint } 6.4561$$

Error from true value: 0.5487.

- Using AIN with symmetric expected values:

$$TAC = [4.1813, 8.7310]_{6.0806}$$

Error: 0.1732 ($3\times$ smaller).

- Using AIN with sample-based expected values:

$$TAC = [4.1813, 8.7310]_{5.9842}$$

Error: 0.0768 ($>7\times$ smaller than CIN).

AINs can significantly reduce error by integrating expected values into interval representation. 18/29

Publication in Fuzzy Sets and Systems

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Fuzzy Sets and Systems

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Asymmetric interval numbers: A new approach to modeling uncertainty

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Asymmintervals library



The screenshot shows the top section of the PyPI page for the `asymintervals` library. It features a search bar with the text "Type '/' to search projects", navigation links for "Help", "Docs", "Sponsors", "Log in", and "Register", and a green "Latest version" badge. Below this, the library name "asymintervals 1.1.1" is displayed in large white text on a blue background. A dark blue button contains the command `pip install asymintervals`. To the right, it says "Released: Nov 18, 2024". At the bottom of this section, it reads "Python Library for Asymmetric Interval Numbers".

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Project description

The `asymintervals` library introduces a novel and unique approach with **Asymmetric Interval Numbers (AINs)**, combining the simplicity of classical interval numbers with advanced capabilities for modeling uncertainty.

AINs integrate the expected value with the interval, offering a more accurate representation of data uncertainty compared to traditional interval numbers. This library provides a complete toolkit, including basic arithmetic operations. The theoretical foundations of AINs, along with detailed discussions on properties, rigorous mathematical proofs, and theorems on symmetry and asymmetry for both binary and unary operations, are introduced in [1], further enhancing the mathematical framework of AINs. Practical examples illustrate the versatility of AINs in various scientific and technical applications. AINs represent a significant advancement in interval arithmetic, paving the way for further research and applications across diverse fields.

Documentation is available on [readthedocs](#).

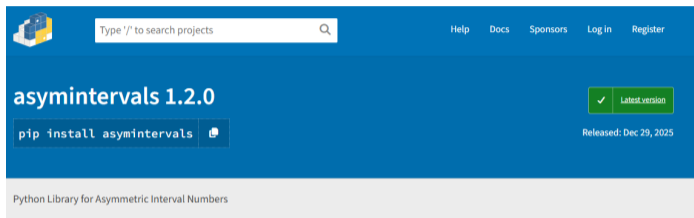
Installation

You can download and install `asymintervals` library using pip:



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Asymmintervals library



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asymmintervals 1.2.0

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Released: Dec 29, 2025

Python Library for Asymmetric Interval Numbers

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Reference

If the `asymintervals` library has contributed to a scientific publication, we kindly request acknowledgment by citing it.

[1] Sałabun, W. (2025). Asymmetric Interval Numbers: a new approach to modeling uncertainty. *Fuzzy Sets and Systems*, 499, 189169. <https://doi.org/10.1016/j.fss.2024.109169>

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@article{sałabun2025,
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  year={2025},
  publisher={Elsevier},
  doi={10.1016/j.fss.2024.109169}}
```

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- Full class description

AIN

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- AIN.__str__()
- AIN.__neg__()
- AIN.__abs__()
- AIN.__add__()
- AIN.__sub__()
- AIN.__mul__()
- AIN.__div__()
- AIN.__pow__()
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- AIN.pdf()
- AIN.cdf()
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- AIN.contains()



Other papers about AIN



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Original software publication

AsymIntervals: A Python library for uncertainty modeling with asymmetric interval numbers

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Modeling Uncertainty in Engineering Problems Using Asymmetric Interval Numbers (AINs)

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ARTICLE INFO	ABSTRACT
<p>Article history: Received 11 August 2025 Received in revised form 22 September 2025 Accepted 23 October 2025 Available online: 27 December 2025</p>	<p>Engineering systems are increasingly complex and subject to multiple sources of uncertainty arising from geometric tolerances, material variability, and operating conditions, which significantly affect performance and reliability, making accurate modeling of uncertainty essential in modern engineering analysis. Classical interval analysis is a well-established approach for representing uncertainty; however, traditional interval numbers (INs) assume a symmetric and uniform distribution around the central value, whereas in practice, uncertainties are often asymmetric due to uneven measurement errors, technological deviations, or differing consequences of overestimation and underestimation. As a result, symmetric intervals may distort risk assessment and reduce the accuracy of results. This study introduces Asymmetric Interval Numbers (AINs), a generalization of classical interval numbers that allows independent modeling of uncertainty in the positive and negative directions. AINs provide a more realistic and flexible framework for representing asymmetric data distributions while retaining the simplicity of interval analysis. To demonstrate their applicability, three case studies involving different classes of engineering problems were conducted, and the results show that AINs improve the realism, stability, and interpretability of uncertainty modeling compared with classical interval approaches, confirming their potential as an effective tool for engineering analysis.</p>
<p>Keywords: Asymmetric Interval Numbers; AINs; Uncertainty Modeling; Engineering Analysis</p>	

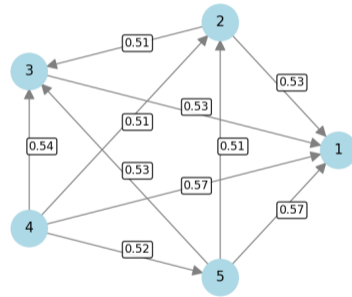
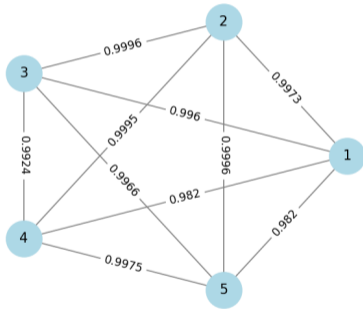


Conclusions and Future Work

AINs = more realistic representation of uncertainty, opening new avenues for decision support.

- Extend AIN:
 - Consider **non-uniform distributions** (beyond uniform).
 - Explore other characteristic values (not only $E[X]$).
- Reduce **overestimation effects** in arithmetic operations.
- Apply AINs in Multi-Criteria Decision Analysis (MCDA):
 - Adapt classical methods currently based on CINs.
 - Improve decision-making under uncertainty.
- Validate on **practical problems**:
 - Engineering, economics, sustainability assessment.
 - Everyday applications.

AIN graphs



Thank you for your attention!

Questions and discussion are welcome.



I invite you to collaboration and further questions.

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